Pergamon

Appl. Math. Lett. Vol. 10, No. 3, pp. 39–42, 1997

0893-9659/97 $17.00 + 0.00

PII: S0893-9659(97)00031-1

Scattering from Stellar Acoustic-Gravity Potentials: II. Phase Shifts via the First Born Approximation

J. A. ADAM
Department of Mathematics and Statistics
Old Dominion University, Norfolk, VA 23529, U.S.A.

I. MCKAIG
Department of Mathematics, Tidewater Community College
Virginia Beach, VA 23456, U.S.A.

(Received August 1996; accepted October 1996)

Abstract—Using the first Born approximation, properties of the scattering phase shift are investigated for waves that are scattered by a schematic representation of a large-scale "stellar potential," i.e., one for which the star itself is viewed as the potential inducing a phase shift in an incoming wave. In particular, the phase shift properties are examined as functions of the relative wavenumber (α) and the azimuthal wavenumber (ℓ), high ℓ-values being of interest in helioseismology.

Keywords—Acoustic-gravity waves, Potential scattering, Phase shifts, Helioseismology.

In two earlier papers [1,2], a scattering problem was posed for waves in stellar interiors by regarding the star itself (assumed nonrotating with no magnetic field) as a large-scale potential well-barrier-well configuration. This particular configuration applies to stars with outer acoustic wave cavity and a possible inner gravity wave cavity, the two being separated by a convectively unstable region through which internal gravity waves are unable to propagate (though tunneling may possibly occur). Standard techniques of potential scattering theory are appropriate to the problem thus posed (details and further references may be found in [2]). Our attention here is focused on the radial partial-wave equation, and the partial-wave solution $u(r)$ which satisfies this governing equation:

$$\frac{d^2u}{dr^2} + \left[k^2 - \frac{\ell(\ell+1)}{r^2} - V(r)\right] u = 0, \tag{1}$$

and in the absence of a potential, the solution $v(r)$ satisfies

$$\frac{d^2v}{dr^2} + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right] v = 0. \tag{2}$$

In each case, the solution is to be regular at the origin and have the following form asymptotically as $r \to \infty$:

$$u(r) \sim \sin\left(kr - \frac{1}{2}\ell\pi + \delta_\ell\right) \tag{3}$$

(where $\delta_\ell = 0$ for the $v(r)$ solution). It follows from (1) and (2) that

$$\int_0^\infty \left\{v\frac{d^2u}{dr^2} - u\frac{d^2v}{dr^2}\right\} dr = \int_0^\infty v(r)V(r)u(r)\,dr. \tag{4}$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Use of the boundary conditions, equation (3), and the fact that the appropriate solution of (2) is

$$v(r) = \left(\frac{\pi k r}{2}\right)^{1/2} J_{\ell+(1/2)}(kr) \tag{5}$$

enables us to write [3]

$$\sin\delta_\ell = -\int_0^\infty \left(\frac{\pi r}{2k}\right)^{1/2} J_{\ell+(1/2)}(kr)V(r)u(r)\,dr, \tag{6}$$

which is an exact integral expression for the phase shift. We now utilize the first Born approximation in this expression, which applies if the potential is in some sense weak. This implies that the distortion of $u(r)$ by the potential will be small, in which case δ_ℓ is small and $u(r) \approx v(r)$. Then it follows that, to this approximation,

$$\delta_\ell = -\frac{\pi}{2}\int_0^\infty rV(r)J^2_{\ell+(1/2)}(kr)\,dr. \tag{7}$$

(An alternative and less direct way of obtaining result (7) may be found in [4].) We now proceed to apply this result to well-barrier-well potentials of the type discussed in [2] (see also [1]), which for simplicity, we write as

$$k^{-2}V(x) = \begin{cases} -V_1, & 0 \le x < a, \\ V_2, & a \le x < b, \\ -V_3, & b \le x < 1, \\ 0, & x \ge 1, \end{cases} \tag{8}$$

where V_i, $i = 1,2,3$, are positive constants, and the stellar radius R_0 has been scaled to unity via the scaled variable $x = r/R_0$. Under these circumstances, and defining $\alpha = kR_0$, equation (7) becomes

$$\delta_\ell = -\frac{\pi R_0^2}{2}\int_0^1 x\,V(x)J^2_{\ell+(1/2)}(\alpha x)\,dx \tag{9}$$

$$= -\frac{\pi\alpha^2}{2} I_\ell(a,b;V_i)\,. \tag{10}$$

We concentrate now on properties of the integral $I_\ell(a,b;V_i)$, using, in particular, the result [5, Section 5.11] that, to within an additive constant,

$$\int^x \xi J_p^2(\alpha\xi)\,d\xi = \frac{x^2}{2}\left[J_p(\alpha x)^2 - J_{p-1}(\alpha x)J_{p+1}(\alpha x)\right]. \tag{11}$$

After some algebra, using form (8) for $V(x)$, it can be shown that

$$\begin{aligned} 2I_\ell(a,b;V_i) &= a^2(V_1+V_2)P(\alpha a) - b^2(V_2+V_3)P(\alpha b) + V_3 P(\alpha) \\ &= -\frac{4}{\pi}\,d_\ell(\alpha), \end{aligned} \tag{12}$$

where $\ell > 0$ and $P(\theta)$ is defined by

$$P(\theta) \equiv J_{\ell-(1/2)}(\theta)J_{\ell+(3/2)}(\theta) - J^2_{\ell+(1/2)}(\theta). \tag{13}$$

The $\ell = 0$ case is directly integrable, and corresponds to s-wave scattering in atomic physics. In the present context, it corresponds to radial modes of oscillation. Since

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x, \tag{14}$$

we have directly from (9) that

$$-\alpha\delta_0 R_0^{-2} = \int_0^1 V(x)\sin^2\alpha x\,dx, \tag{15}$$

whence, after some algebra

$$\begin{aligned} 2\delta_0\alpha^{-1} &= a(V_1+V_2)Q(\alpha a) - b(V_2+V_3)Q(\alpha b) + V_3 Q(\alpha) \\ &= d_0(\alpha), \end{aligned} \tag{16}$$

where

$$Q(\theta) \equiv 1 - \frac{\sin 2\theta}{2\theta}. \tag{17}$$

As $\alpha \to \infty$,

$$2\delta_0\alpha^{-1} \to [V_1 a - V_2(b-a) + V_3(1-b)], \tag{18}$$

and as $\alpha \to 0^+$,

$$3\delta_0 \sim \alpha^3\left[V_1 a^3 - V_2\left(b^3-a^3\right) + V_3\left(1-b^3\right)\right] + 0\left(\alpha^5\right). \tag{19}$$

Note that in the case of a single potential well of depth V_0 in [0,1], from (9),

$$\delta_0 = \frac{\alpha V_0}{2}\left(1 - \frac{\sin 2\alpha}{2\alpha}\right) = \frac{\alpha V_0}{2}Q(\alpha), \tag{20}$$

which is just (16) with $a = b = 0$ and $V_3 = V_0$.

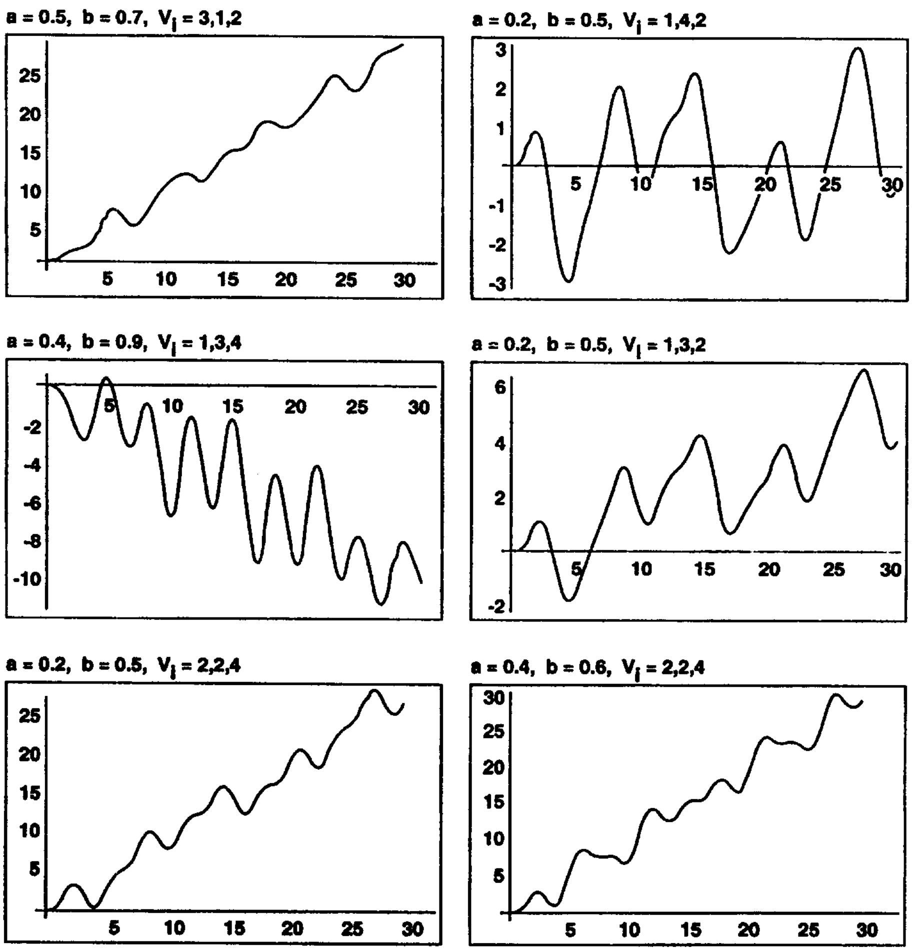

Figure 1. $(\alpha/2)d_0(\alpha) = \delta_0(\alpha)$ for various values of a, b, and V_i.

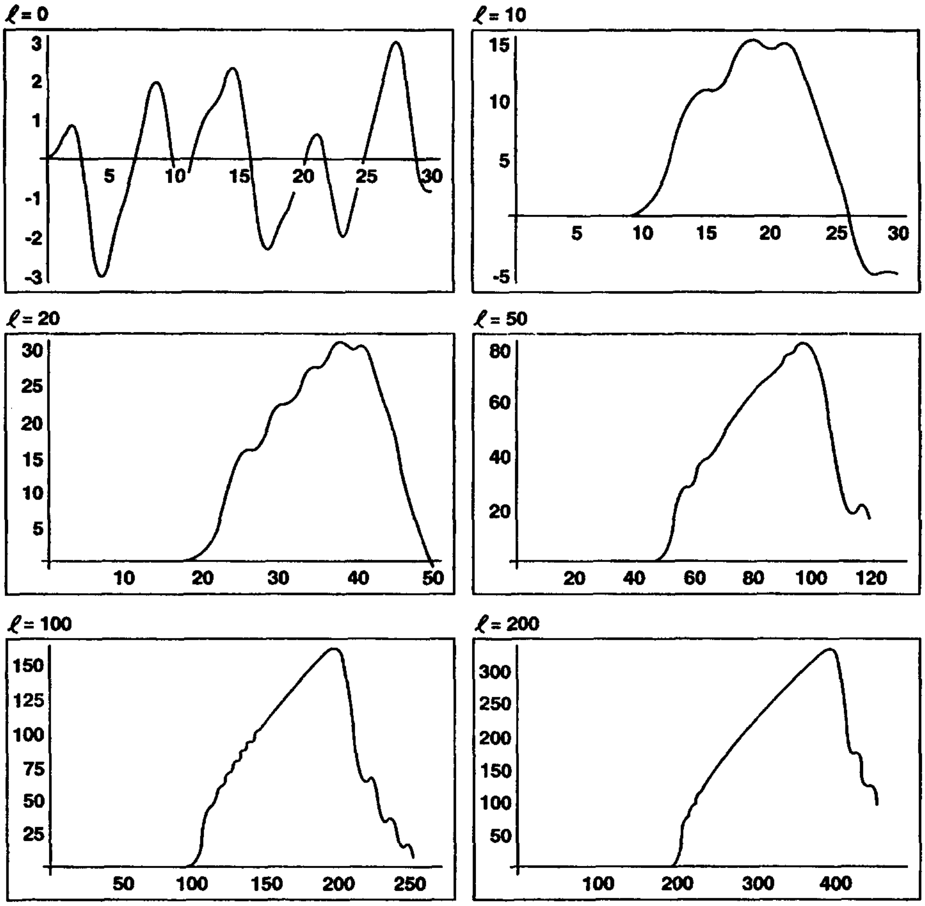

Figure 2. $\alpha^2 d_\ell(\alpha) = \delta_\ell(\alpha)$ for various $\ell \in [0, 200]$.

In Figure 1, the $\ell = 0$ case is depicted for various combinations of the V_i (in units of k^2) and a and b. The quantity plotted is $(\alpha/2)d_0(\alpha) = \delta_0(\alpha)$ from (16). In Figure 2, the corresponding graphs of $\alpha^2 d_\ell(\alpha) = \delta_\ell(\alpha)$ are shown for various values of ℓ (including the limit as $\ell \to 0+$). Noting that $\delta_\ell(k)$ has an inverse tangent behavior as a function of k^2 [3] wherein the maximum slope occurs at a resonance, we may infer from these figures the existence of several resonances for each of the $\ell = 0$ cases. This behavior is more striking still for $\ell \neq 0$. The steeper the slope is, the sharper is the resonance, and interestingly, these inferred resonances appear to exist at those α for which $k \approx \ell/R_0$, i.e., at radial length scales L inversely proportional to ℓ. For the sun, with $R_0 \approx 7 \times 10^5$ km and $\ell \approx 200$, this corresponds to $L \sim k^{-1} \approx 3000$ km. An examination of the asymptotic behavior in ℓ of $\frac{\partial \delta_\ell}{\partial k}$ may provide more insight into the location of these resonances. This will be deferred until a later study.

REFERENCES

1. J.A. Adam, The scattering potential for a polytrope of degree 5, *Appl. Math. Lett.* **6** (4), 9–12 (1993).
2. J.A. Adam, Non-radial stellar oscillations: A perspective from potential scattering. (I). Theoretical foundations, *Astrophys. Sp. Sci.* **220**, 179 (1994).
3. P.G. Burke, *Potential Scattering in Atomic Physics*, Plenum Press, New York, (1977).
4. N.F. Mott and H.S.W. Massey, *Theory of Atomic Collisions*, Clarendon Press, Oxford, (1965).
5. G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, (1966).